

The Itô exponential on Lie Groups

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Abstract

Let G be a Lie Group with a left invariant connection ∇^G . Denote by \mathfrak{g} the Lie algebra of G , which is equipped with a connection $\nabla^{\mathfrak{g}}$. Our main is to introduce the concept of the Itô exponential and the Itô logarithm, which take in account the geometry of the Lie group G and the Lie algebra \mathfrak{g} . This definition characterize directly the martingales in G with respect to the left invariant connection ∇^G . Further, if any $\nabla^{\mathfrak{g}}$ geodesic in \mathfrak{g} is send in a ∇^G geodesic we can show that the Itô exponential and the Itô logarithm are the same that the stochastic exponential and the stochastic logarithm due to M. Hakim-Dowek and D. Lépingle in [11]. Consequently, we have a Campbell-Hausdorff formula. From this formula we show that the set of affine maps from (M, ∇^G) into (G, ∇^G) is a subgroup of the Loop group [16]. As in general, the Lie algebra is considered as smooth manifold with a flat connection, we show a Campbell-Hausdorff formula for a flat connection on \mathfrak{g} and a bi-invariant connection on G . To this main we introduce the definition of the null quadratic variation property. To end, we use the Campbell-Hausdorff formula to show that a product of harmonic maps with value in G is a harmonic map.

Key words: Lie groups; Exponential map; Campbell-Hausdorff formula; stochastic analysis on manifolds

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1 Introduction

M. Hakim-Dowek and D. Lpingle introduced by first time a concept of exponential stochastic on general Lie Groups in [11]. Their idea can be interpreted in the following way. Let G be a Lie group and \mathfrak{g} its Lie algebra. Given a semimartingale $M \in \mathfrak{g}$, the exponential stochastic $X = e(M)$ is a solution of the stochastic differential equation, in the Stratonovich sense,

$$\delta X = L_{X*} \delta M, X_0 = e.$$

Furthermore, they showed a existence of the inverse of the stochastic exponential, which is the stochastic logarithm. In this paper [11] was developed a serial of result about both the stochastic exponential and the stochastic logarithm. We emphasize the Campbell-Hausdorff formula.

The concept of the stochastic exponential and the stochastic logarithm on Lie group has been studied and applied in some situations. For example, M. Arnaudon developed studies of exponential stochastic in Lie groups in the case that G has a left invariant connection [2]. He also used the stochastic exponential to study the martingales and Brownian motions in homogeneous space [3]. A

characterization of semimartingales, martingales and Brownian motions on a principal fiber bundle due to M. Arnaudon and S. Paycha is obtained with stochastic exponential in [4].

Let G be a Lie group with a bi-invariant metric. P. Catuogno and P. Ruffino in [5] used the stochastic exponential and the stochastic logarithm to show that the product of harmonic maps with values in G is a harmonic map.

The developed of the stochastic exponential has been done without to takes in account a geometry of Lie group. Despite of the above studies have worked with some types of connections. This fact occurs because the integral of Stratonovich do not has intrinsically the geometry of the smooth manifolds. Unlike, the Itô integral in a smooth manifolds has intrinsically the information of geometry of smooth manifolds. In this way, we introduce a stochastic exponential and a stochastic logarithm in the Itô sense. Let ∇^G and $\nabla^{\mathfrak{g}}$ be connections on G and \mathfrak{g} , respectively. The Itô exponential and the Itô logarithm are, respectively, the solutions of the following stochastic differential equations

$$d^G X = L_{X*} d^{\mathfrak{g}} M, \quad X_0 = e$$

and

$$d^{\mathfrak{g}} N = L_{Y^{-1}*} d^G Y, \quad Y_0 = 0.$$

The meaning of these solutions are given in (6) and (8).

Our first work is to show that this equations have unique solutions, which do not explode in a finite time if ∇^G is a left invariant connection. Also, we show that the operators $e^{G^{\mathfrak{g}}}$ and $\mathcal{L}^{G^{\mathfrak{g}}}$ are inverses. As direct consequence we get that every ∇^G -martingales is given by $e^{G^{\mathfrak{g}}}(M)$ for a $\nabla^{\mathfrak{g}}$ -martingale M in \mathfrak{g} . Despite of the Itô exponential and the Itô logarithm being dependent of the connections ∇^G and $\nabla^{\mathfrak{g}}$, it is not necessary a correspondence between these connections. However, ∇^G is a left invariant connection, and so there exists a unique bilinear form α on \mathfrak{g} associated to ∇^G . We use α to construct a $\nabla^{\mathfrak{g}}$ connection on \mathfrak{g} . Then, ∇^G and $\nabla^{\mathfrak{g}}$ have a deep correspondence. In fact, the geodesics in \mathfrak{g} are sent to the geodesics in G and vice versa. With this fact we show that $e^{G^{\mathfrak{g}}}(M)$ and $\mathcal{L}^{G^{\mathfrak{g}}}(X)$ are the stochastic exponential and stochastic logarithm in the Stratonovich sense. Consequently, we have the Campbell-Hausdorff formula for semimartingales. Using Campbell-Hausdorff formula, in this environment, we show that the product of the Lie Group is an affine map. According with this result, a product of two affine maps with values in the Lie group G is an affine map. Further, we prove that if $F : (M, \nabla^M) \rightarrow (G, \nabla^G)$ is an affine map, then the application F^{-1} , defined by $F^{-1}(x) = (F(x))^{-1}$, is also an affine map. As direct consequence, the set of affine applications from a smooth manifold (M, ∇^M) into the Lie group (G, ∇^G) is a subgroup of the Loop Group (see for instant [16]).

In general, the Lie algebra \mathfrak{g} is only consider as vector space. In other words, \mathfrak{g} is a smooth manifold endowed with a flat connection. Thinking in this context, we prove a Campbell- Hausdorff formula without to assume some correspondence between $\nabla^{\mathfrak{g}}$ and a left invariant connection ∇^G . Despite the Itô exponential and Itô logarithm allow this generality, to prove the Campbell-Hausdorff formula we need to assume two hypothesis. First, the connection ∇^G is a bi-invariant connection. Second, we need that the semimartingales satisfy the null quadratic variation property. It means that two semimartingales X, Y in a smooth manifold have the null quadratic variation property if for every

local coordinate system (U, x_1, \dots, x_n) we have $[X^i, Y^j] = 0$, where $X^i = x_i \circ X$ and $Y^j = x_j \circ Y$ for $i, j = 1, \dots, n$. To end, we can show that a product of two harmonic maps with values in G , which has a bi-invariant connection, is a harmonic map.

2 Preliminaries

In this work we use freely the concepts and notations of P. Protter [17], E. Hsu [13], P. Meyer [15], M. Emery [8] and [10], and S. Kobayashi and N. Nomizu [14]. We suggest the reading of [6] for a complete survey about the objects of this section. From now on the adjective smooth means C^∞ .

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a probability space which satisfies the usual hypotheses (see for example [8]). Our basic assumption is that every stochastic process is continuous.

Let M be a smooth manifold and X_t a continuous stochastic process with values in M . We call X_t a semimartingale if, for all f smooth function, $f(X_t)$ is a real semimartingale.

Let M be a smooth manifold endowed with a connection ∇^M . Let X be a semimartingale in M and θ a 1-form on M defined along X . Let (U, x_1, \dots, x_n) be a local coordinate system on M . We define the Stratonovich and Itô integrals, respectively, of θ along X , locally, by

$$\int_0^t \theta \delta X_s = \int_0^t \theta_i(X_s) dX_s^i + \frac{1}{2} \int_0^t \frac{\partial \theta_i}{\partial x^j}(X_s) d[X^i, X^j]_s,$$

and

$$\int_0^t \theta d^{\nabla^M} X_s = \int_0^t \theta_i(X_s) dX_s^i + \frac{1}{2} \int_0^t \Gamma_{jk}^i(X_s) \theta_i(X_s) d[X^j, X^k]_s, \quad (1)$$

where $\theta = \theta_i dx^i$ with θ_i smooth functions and Γ_{jk}^i are the Christoffel symbols of the connection ∇^M . Let $b \in T^{(2,0)}M$ be defined along X . We define the quadratic integral on M along X , locally, by

$$\int_0^t b(dX, dX)_s = \int_0^t b_{ij}(X_s) d[X^i, X^j]_s,$$

where $b = b_{ij} dx^i \otimes dx^j$ with b_{ij} smooth functions.

A direct consequence of the definitions above is the Stratonovich-Itô formula of conversion given by

$$\int_0^t \theta \delta X_s = \int_0^t \theta d^{\nabla^M} X_s + \frac{1}{2} \int_0^t \nabla^M \theta (dX, dX)_s. \quad (2)$$

A semimartingale X with values in M is called a ∇^M -martingale if $\int \theta d^{\nabla^M} X$ is a real local martingale for all $\theta \in \Gamma(TM^*)$.

Let M be a Riemannian manifold with a metric g . A semimartingale B in M is said a g -Brownian motion if B is a ∇^g -martingale, being ∇^g the Levi-Civita connection of g , and for any section b of $T^{(2,0)}M$ we have

$$\int_0^t b(dB, dB)_s = \int_0^t \text{tr } b(B_s) ds. \quad (3)$$

Let M and N be manifolds, θ be a section of TN^* , b be a section of $T^{(2,0)}N$ and $F : M \rightarrow N$ be a smooth map. For a semimartingale X_t in M , we have the following Itô formula for Stratonovich integral:

$$\int_0^t \theta \delta F(X) = \int_0^t F^* \theta \delta X. \quad (4)$$

Let M and N be smooth manifold with connections ∇^M and ∇^N . The geometric Itô formula is given by

$$\int_0^t \theta d^{\nabla^N} F(X_s) = \int_0^t F^* \theta d^{\nabla^M} X_s + \frac{1}{2} \int_0^t \beta_F^* \theta (dX, dX)_s. \quad (5)$$

A useful result for us is Proposition 7.8 in the book [8]. How we use many times we state it here for convenient of the reader.

Proposition 2.1 *Let M be a smooth manifold endowed with a connection ∇^M , X a semimartingale in M , K a predictable, locally bounded, real process and f a smooth on M . Then, for all 1-form θ along X , the Itô integral has the following properties:*

- (i) $\int (K\theta) d^{\nabla^M} X = \int K d \int \theta d^{\nabla^M} X;$
- (ii) $\int df d^{\nabla^M} X = f \circ X - f \circ X_0 - \frac{1}{2} \int \text{Hess}^M f(dX, dX).$

3 Itô Exponential and Itô Logarithm

Let G be a Lie Group and \mathfrak{g} its Lie algebra. Let us denote by L_g the left translation on G . From this we can construct the following family of linear applications on $\mathfrak{g}^* \otimes TG$: since \mathfrak{g} is isomorphic to $T\mathfrak{g}$, we consider that the left translation is a linear application $L_{g*}(e) : \mathfrak{g} \rightarrow TG$, for all $g \in G$. We observe that the family of the applications $L_{g*}(e) : \mathfrak{g} \rightarrow TG$ is smooth in the following sense. Taking $E \in \mathfrak{g}$ we obtain a smooth left invariant vector field $X \in TG$ such that $L_{g*}(e)(E) = X_g$. Therefore $L_{g*}(e)$ is a smooth family from $\mathfrak{g} \times G$ into TG (see for instant Definition 6.34 in [8]).

We endow G with a left-invariant connection ∇^G and \mathfrak{g} with a connection $\nabla^{\mathfrak{g}}$. Let X be a semimartingale in G and M a semimartingale in \mathfrak{g} . One says that X is a solution to the Itô stochastic differential equation

$$d^{\nabla^G} X_t = L_{(X_t)*}(e) d^{\nabla^{\mathfrak{g}}} M, \quad (6)$$

if, for every 1-form θ on G , the real semimartingales $\int \theta d^{\nabla^G} X$ and $\int L_{X_t}^*(e) \theta d^{\nabla^{\mathfrak{g}}} M$ are equals.

Theorem 3.1 *Let M be a semimartingale on \mathfrak{g} and X_0 a \mathcal{F}_0 -measurable random variable on G . There exist a predictable stopping time ζ and a G -valued semimartingale X in G on the interval $[0, \zeta[$, with initial condition X_0 , solution to (6) and exploding to times ζ on the event $\{\zeta < \infty\}$. Moreover the following uniqueness and maximality properties holds: if ζ' is a predictable time and X' a solution starting from X_0 defined on $[0, \zeta'[$, then $\zeta' \leq \zeta$ and $X' = X$ on $[0, \zeta'[$.*

Proof: From the Itô transfer principle, see Theorem 12 in [9], Itô stochastic differential equation (6) is equivalent to the intrinsic equation $d^2X = f(X, M)d^2M$, where $f : \tau_{M_t(\omega)}\mathfrak{g} \rightarrow \tau_{X_t(\omega)}G$ is the unique semi-affine Schwartz morphism with $L_{X_t\omega^*}(e)$ as restriction to the first order. From Lemma 11 in [9] we see that f is a family of Schwartz morphism which depend smoothly upon (e, g) , for all $g \in G$. From Theorem 6.41 in [8], with a \mathcal{F}_0 -measurable random variables X_0 in G as an initial condition, there exists a predictable stopping time $\zeta > 0$ and a G -valued semimartingale X on the interval $[0, \zeta[$, with initial value X_0 , solution to $d^2X = f(X, M)d^2M$ and exploding at time ζ on the event $\{\zeta < \infty\}$. Moreover the uniqueness and maximality properties holds in the sense that if ζ' is a predictable time and X' a solution starting from X_0 defined on $[0, \zeta'[$, then $\zeta' \leq \zeta$ and $X' = X$ on $[0, \zeta'[$. Again, Itô transfer principle assure the unique and maximal solution of (6) with initial condition X_0 and predictable stopping time ζ .

Next result justifies the necessity of we work with left-invariant connections on G . We observe that Theorem above is hold with hypothesis that ∇^G is any connection on G .

Proposition 3.2 *Let G be a Lie group and \mathfrak{g} its Lie algebra. Assume that ∇^G is a left-invariant connection on G and $\nabla^{\mathfrak{g}}$ is a connection on \mathfrak{g} . Suppose that Y_t is a solution of (6) with initial condition $Y_0 = e$. If X_0 is \mathcal{F}_0 -measurable random measure on G , then $X_t = X_0Y_t$ is also a solution of (6).*

Proof: We begin denoting the product on Lie group G by m . Let θ be a 1-form on G . As a function to m , the Itô integral along X_t is writing as

$$\int \theta d^{\nabla^G} X_t = \int \theta d^{\nabla^G} X_0 Y_t = \int \theta d^{\nabla^G} m(X_0, Y_t).$$

The geometric Itô formula (5) get

$$\int \theta d^{\nabla^G} X_t = \int m^* \theta d^{\nabla^G \times \nabla^G} (X_0, Y_t) + \frac{1}{2} \int \beta_m^* \theta (d(X_0, Y_t), d(X_0, Y_t)).$$

From Proposition 3.15 in [10] we see that

$$\int \theta d^{\nabla^G} X_t = \int (R_{Y_t}^* \theta) d^{\nabla^G} X_0 + \int (L_{X_0}^* \theta) d^{\nabla^G} Y_t + \frac{1}{2} \int \beta_m^* \theta (d(X_0, Y_t), d(X_0, Y_t)).$$

X_0 is a constant process, and consequently

$$\int \theta d^{\nabla^G} X_t = \int (L_{X_0}^* \theta) d^{\nabla^G} Y_t + \frac{1}{2} \int \beta_m^* \theta (d(X_0, Y_t), d(X_0, Y_t)).$$

We claim that the $\beta_m(d(X_0, Y_t), d(X_0, Y_t))$ is null. In fact, let $0 \in T_g G$ and Y a left invariant vector field on G . Here, 0 is the vector associated to the constant process X_0 . Then

$$\begin{aligned} \beta_m(0, Y) &= \nabla_{m_*(0, Y)}^G m_*(0, Y) - m_* \nabla^{G \times G}(0, Y) \\ &= \nabla_{R_{h*}0 + L_{g*}(Y)}^G (R_{h*}0 + L_{g*}(Y)) - m_* \nabla^{G \times G}(0, Y) \\ &= \nabla_{L_{g*}Y}^G L_{g*}Y - L_{g*}(\nabla_Y^G Y) \\ &= L_{g*}(\nabla_Y^G Y) - L_{g*}(\nabla_Y^G Y) \\ &= 0, \end{aligned}$$

where in forth equality we use the fact that ∇^G is a left invariant connection. Thus we get

$$\int \theta d^{\nabla^G} X_t = \int (L_{X_0}^* \theta) d^{\nabla^G} Y_t.$$

As Y_t is a solution of (6) we have

$$\int \theta d^{\nabla^G} X_t = \int L_{Y_t}^*(e) L_{X_0}^*(Y_t) \theta d^{\nabla^g} M_t.$$

This gives

$$\int \theta d^{\nabla^G} X_t = \int L_{X_t}^*(e) \theta d^{\nabla^g} M_t.$$

Therefore we conclude that X_t is a solution of (6).

Proposition 3.3 *Let X_t be the solution of stochastic equation (6) then this life time is infinity.*

Proof: Let X_t the solution of (6) with initial condition $X_0 = e$. Then there exist a predictable stopping time ζ such that X_t is defined for $t \in [0, \zeta]$. Define

$$X_t = X_{(n\zeta)} X_{(t-n\zeta)}, \quad n\zeta \leq t \leq (n+1)\zeta$$

Using Proposition 3.2 we conclude that X_t is solution of (6) for $n\zeta \leq t \leq (n+1)\zeta$ because $X_{t-n\zeta}$ is. Therefore X_t is defined for all $t \geq 0$.

Proposition 3.2 says that we can consider the solution of stochastic differential equation (6) with initial value $X_0 = e$ rather than any random variable on G . In other side, Proposition 3.3 shows that solution of (6) is in interval $[0, \infty[$. From these facts we give the following definition.

Definition 3.1 *We will denote by $e^{G^g}(M)$ the solution of (6) with initial condition $X_0 = e$ and we call it Itô stochastic exponential with respect to ∇^G and ∇^g .*

In the follow, for simplicity, we will call $e^{G^g}(M)$ by Itô exponential.

Remark 1 It is well known that the left-invariant connections on G are in one-one correspondence with bilinear forms on \mathfrak{g} , see Proposition 1, chapter 3 in [12]. However the stochastic differential equation (6) do not preserve this fact, that is, it is not necessary that ∇^G and ∇^g have some association.

As an immediate consequence from Theorem 3.1 we have a characterization of the ∇^G -martingales on Lie group G .

Corollary 3.4 *The Itô exponential $e^{G^g}(M)$ is a ∇^G -martingale on G if and only if M is a ∇^g -martingale in \mathfrak{g} .*

The Itô exponential yields a semimartingale in G from a semimartingale in \mathfrak{g} . M. Hakim-Dowek and D. Lépingle [11] define, in Stratonovich sense, an inverse of the exponential stochastic, which they called the stochastic logarithm. We desire to get an analogous one in Itô sense. For this we consider a left-invariant connection ∇^G on G and a connection ∇^g on \mathfrak{g} . Our idea is create the process

inverse of Itô exponential as solution of the following Itô stochastic differential equation

$$d^{\nabla^{\mathfrak{g}}} M_t = L_{(X_t)^{-1}*}(X_t) d^{\nabla^G} X_t. \quad (7)$$

The solution of this differential equation means that for every 1-form ψ on \mathfrak{g} , the real semimartingales $\int \psi d^{\nabla^{\mathfrak{g}}} M$ and $\int L_{X_t}^*(X_t) \psi d^{\nabla^G} X$ are equal. The solution and uniqueness are assured to follow.

Theorem 3.5 *Let X be a semimartingale on G and M_0 a \mathcal{F}_0 -measurable random variable on \mathfrak{g} . There exists a predictable stopping time η and a \mathfrak{g} -valued semimartingale M in \mathfrak{g} on the interval $[0, \eta[$, with initial condition M_0 , solution to (8) and exploding to times η on the event $\{\eta < \infty\}$. Moreover the following uniqueness and maximality properties holds: if η' is a predictable time and M' a solution starting from M_0 defined on $[0, \eta']$, then $\eta' \leq \eta$ and $M' = M$ on $[0, \eta']$.*

Proof: The proof is analogous to one in Theorem 3.1.

Next, we show that it is possible translate the X_t to begin at origin of Lie Group without to change the solution of (8). From this fact we will consider only solutions of (8) with initial condition $M_0 = 0$.

Proposition 3.6 *Let G be a Lie group and \mathfrak{g} its Lie algebra. Assume that ∇^G is a left-invariant connection on G and $\nabla^{\mathfrak{g}}$ is a connection on \mathfrak{g} . Suppose that M_t is a solution of (8) with respect to a semimartingale Y_t and the initial condition $M_0 = 0$. If X_0 is a \mathcal{F}_0 -measurable random measure on G , then M_t is a solution of (8) with respect to $X_t = X_0 Y_t$.*

Proof: Let θ be a 1-form on \mathfrak{g} . By definition, the Itô stochastic differential equation (8) means that

$$\int \theta d^{\nabla^{\mathfrak{g}}} M_t = \int L_{Y_t}^* \theta d^{\nabla^G} Y_t.$$

Since $Y_t = L_{X_0}^{-1} X_t$, it follows that

$$\int \theta d^{\nabla^{\mathfrak{g}}} M_t = \int L_{(X_0^{-1} X_t)^{-1}}^* \theta d^{\nabla^G} L_{X_0}^{-1} X_t.$$

As in the proof of Proposition 3.2, the equation above gives

$$\int \theta d^{\nabla^{\mathfrak{g}}} M_t = \int L_{X_0}^* L_{(X_0^{-1} X_t)^{-1}}^* \theta d^{\nabla^G} X_t.$$

Thus we get

$$\int \theta d^{\nabla^{\mathfrak{g}}} M_t = \int L_{X_t}^* \theta d^{\nabla^G} X_t.$$

By definition, M_t is also a solution of (8) with respect to X_t .

Proposition 3.7 *The solutions of (8) have life time infinity.*

Proof: Let M_t be the semimartingale on \mathfrak{g} solution of (8) in the interval $[0, \eta]$. For each positive integer n we consider the random variable $X_{n\eta}$. Taking the process $Y_t = X_{n\eta} X_t$ the Proposition (3.6) assures that M_t satisfies

$$d^{\nabla^{\mathfrak{g}}} M_t = L_{(Y_t)^{-1}*}(Y_t) d^{\nabla^G} Y_t, \quad (8)$$

for $t \in [n\eta, (n+1)\eta[$. Therefore M_t is defined in $[0, \infty[$.

Theorem 3.5 and Propositions 3.6 and 3.7 yield the good definition of Itô stochastic Logarithm.

Definition 3.2 *The solution of (8), with initial condition $M_0 = 0$, is called Itô logarithm with respect to ∇^G and $\nabla^{\mathfrak{g}}$ and it is denoted by $\mathcal{L}^{G\mathfrak{g}}(X)$.*

It is clear the relation about $\nabla^{\mathfrak{g}}$ -martingales and ∇^G -martingales.

Corollary 3.8 *A semimartingale X_t in G is a ∇^G -martingale if and only if $\mathcal{L}^{G\mathfrak{g}}(X)$ is $\nabla^{\mathfrak{g}}$ -martingale.*

Our intention in construct Itô Logarithm is that it be the inverse of Itô exponential. We will show that it is true.

Theorem 3.9 *Let X_t, M_t be semimartingales on G and \mathfrak{g} , respectively. Then*

$$\mathcal{L}^{G\mathfrak{g}}(e^{G\mathfrak{g}}(M_t)) = M_t$$

and

$$e^{G\mathfrak{g}}(\mathcal{L}^{G\mathfrak{g}}(X_t)) = X_t$$

Proof: Let ψ be a 1-form on \mathfrak{g} and M_t a semimartingale on \mathfrak{g} . The Itô exponential is the semimartingale $e^{G\mathfrak{g}}(M_t)$ on G . By definition, Itô logarithm apply to $e^{G\mathfrak{g}}(M_t)$ means that

$$\int \psi d^{\nabla^{\mathfrak{g}}} \mathcal{L}^{G\mathfrak{g}}(e^{G\mathfrak{g}}(M_t)) = \int L_{e^{G\mathfrak{g}}(M_t)}^* \psi d^{\nabla^G} e^{G\mathfrak{g}}(M_t)$$

Since $e^{G\mathfrak{g}}(M_t)$ is solution of (6), it follows that

$$\int \psi d^{\nabla^{\mathfrak{g}}} \mathcal{L}^{G\mathfrak{g}}(e^{G\mathfrak{g}}(M_t)) = \int L_{e^{G\mathfrak{g}}(M_t)}^* L_{e^{G\mathfrak{g}}(M_t)}^{-1*} \psi d^{\nabla^{\mathfrak{g}}} M_t = \int \psi d^{\nabla^{\mathfrak{g}}} M_t$$

As ψ is an arbitrary 1-form on \mathfrak{g} we have $\mathcal{L}^{G\mathfrak{g}}(e^{G\mathfrak{g}}(M_t)) = M_t$. Similarly, for a semimartingale X_t on G , we get that $e^{G\mathfrak{g}}(\mathcal{L}^{G\mathfrak{g}}(X_t)) = X_t$.

Theorem 3.10 *Every ∇^G -martingale on G is write as $e^{G\mathfrak{g}}(M_t)$ for a $\nabla^{\mathfrak{g}}$ -martingale M_t on \mathfrak{g} .*

Proof: Let X_t be a ∇^G -martingale on G . From Corollary 3.8 we see that $\mathcal{L}^{G\mathfrak{g}}(X_t)$ is a $\nabla^{\mathfrak{g}}$ -martingale. Taking $M_t = \mathcal{L}^{G\mathfrak{g}}(X_t)$ we conclude that $X_t = e^{G\mathfrak{g}}(M_t)$, which follows from Theorem 3.9.

4 Campbell-Hausdorff formulas- first situation

Let G be a Lie group and \mathfrak{g} its Lie algebra. Let ∇^G be a left invariant connection on G . We know that there exist a unique bilinear α of $\mathfrak{g} \times \mathfrak{g}$ into \mathfrak{g} associated to ∇^G . We begin constructing an specific connection on \mathfrak{g} .

Proposition 4.1 *For every bilinear application $\alpha : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ there exists only one connection $\nabla^{\mathfrak{g}}$ associated to α .*

Proof: Let $\{E_1, \dots, E_n\}$ be a basis in \mathfrak{g} . We define the connection $\nabla^{\mathfrak{g}}$ by

$$\nabla_{E_i}^{\mathfrak{g}} E_j = \alpha(E_i, E_j).$$

We extend in the usual way $\nabla_X^{\mathfrak{g}} Y$ for X, Y in \mathfrak{g} .

Here and subsequently, we call the connection $\nabla^{\mathfrak{g}}$ given by Proposition 4.1 of the connection associated to the left invariant connection ∇^G .

According to initial observation, every left invariant connection ∇^G on G has an associated bilinear form $\alpha : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ and vice versa. Being the Christoffel symbols of ∇^G at origin e and one of $\nabla^{\mathfrak{g}}$ it is expected a relation between both. For us this relation is very important and its first consequence is given below.

Lemma 4.2 *Let G be a Lie group and \mathfrak{g} its Lie algebra. Suppose that G has a left invariant connection ∇^G and \mathfrak{g} is endowed with its associated connection $\nabla^{\mathfrak{g}}$. If $x(t)$ is a $\nabla^{\mathfrak{g}}$ -geodesic in \mathfrak{g} , then the solution of differential equation*

$$\dot{y}(t) = L_{y_t*}(e)\dot{x}(t), \quad y_0 = g,$$

is a $\nabla^{\mathfrak{g}}$ -geodesic on G .

Proof: To prove the Lemma it is only necessary to observe that

$$\begin{aligned} \nabla_{\dot{y}(t)}^G \dot{y}(t) &= \nabla_{L_{y_t*}(e)\dot{x}(t)}^G L_{y_t*}(e)\dot{x}(t) \\ &= L_{y_t*}(e)(\nabla_{\dot{x}(t)}^G \dot{x}(t))(e) \\ &= L_{y_t*}(e)(\nabla_{\dot{x}(t)}^{\mathfrak{g}} \dot{x}(t))(e) \\ &= 0 \end{aligned}$$

because ∇^G is a left invariant connection.

M. Emery has observed in [9], in a general context, the utility of Lemma 4.2. From this he proved an equivalence between Stratonovich and Itô differential equations. So, we use Lemma 4.2 to show that the Itô and Stratonovich exponentials are the same.

Theorem 4.3 *Let G be a Lie group and \mathfrak{g} its Lie algebra. Suppose that G has a left invariant connection ∇^G and \mathfrak{g} is endowed with its associated connection $\nabla^{\mathfrak{g}}$. If M_t is a semimartingale in \mathfrak{g} , then the Itô and Stratonovich stochastic differential equations*

$$\begin{aligned} d^{\nabla^G} X_t &= L_{(X_t)*}(e) d^{\nabla^{\mathfrak{g}}} M_t, \\ \delta X_t &= L_{(X_t)*}(e) \delta M_t \end{aligned}$$

are equivalents.

Proof: First we observe that the family of the linear applications $L_{g*}(e) : \mathfrak{g} \rightarrow TG$, $g \in G$, is C^∞ . The proof comes from Lemma 4.2. In fact, it is only necessary to see that Lemma 4.2 assures the hypothesis of Corollary 16 in [9] are true. Consequently, the proof follows.

Let M_t be a semimartingale in \mathfrak{g} . M. Hakim-Dowek and D. Lépingle [11] define the Stratonovich stochastic exponential $X_t = e(M_t)$ in G as a solution of the stochastic differential equation

$$\delta X_t = L_{(X_t)*}(e) \delta M_t, \quad X_0 = e.$$

In the case that X_t is a semimartingale in G , the Stratonovich stochastic logarithm is given by solution of the stochastic differential equation

$$\delta M_t = L_{(X_t)^{-1}*}(X_t)\delta X_t, M_0 = 0.$$

Combining these definitions with Theorem 4.3 yields the equalities between exponential and logarithm in the Itô and Stratonovich senses.

Corollary 4.4 *Let G be a Lie group and \mathfrak{g} its Lie algebra. Suppose that G has a left invariant connection ∇^G and \mathfrak{g} is endowed with its associated connection $\nabla^{\mathfrak{g}}$. The following statement are true:*

1. *For each semimartingale M_t in \mathfrak{g} , $e^{G\mathfrak{g}}(M_t) = e(M_t)$.*
2. *For each semimartingale X_t in G , $\mathcal{L}^{G\mathfrak{g}}(X_t) = \mathcal{L}(X_t)$.*

Proof: The first affirmative is a direct consequence of Theorem 4.3. The second is a consequence of item 1. and Theorem 5 in [11] for Stratonovich Logarithm. Indeed, we have

$$e(\mathcal{L}^{G\mathfrak{g}}(X_t)) = e^{G\mathfrak{g}}(\mathcal{L}^{G\mathfrak{g}}(X_t)) = X_t = e(\mathcal{L}(X_t)),$$

which gives $\mathcal{L}^{G\mathfrak{g}}(X_t) = \mathcal{L}(X_t)$.

In the case of the Stratonovich exponential and logarithm we have an stochastic Campbell-Hausdorff formulas (see Theorem in [11]). To simplify the reading we cite those formulas here. Let X, Y be semimartingales on G and M, N semimartingales on \mathfrak{g} , it holds that

$$\mathcal{L}(X_t Y_t) = \int Ad(Y_t^{-1})\delta\mathcal{L}(X_t) + \mathcal{L}(Y_t) \quad (9)$$

$$e(M + N) = e\left(\int Ad(e(N))\delta M\right)e(N). \quad (10)$$

Before we proceed our work we need to introduce some notations. Let $\{E_1, \dots, E_n\}$ be a basis on \mathfrak{g} . The system of coordinates associated to this basis is denoted by (x_1, \dots, x_n) . Thus, given a semimartingale M in \mathfrak{g} we write $M = M^i E_i$, where $M^i = x^i \circ M$ are real semimartingales for $i = 1, \dots, n$. For our purpose, taking in account two semimartingales M, N in \mathfrak{g} we adopted the following notations

$$\int Ad(e(N))\delta M = \sum_{i=1}^n \int Ad(e(N))(E_i)\delta M^i$$

and

$$\int Ad(e^{G\mathfrak{g}}(N))dM = \sum_{i=1}^n \int Ad(e^{G\mathfrak{g}}(N))(E_i)dM^i. \quad (11)$$

Since our purpose is to show a Hausdorff-Campbell formula for the Itô exponential and logarithm, the next step is to see that both integral above are the same.

Lemma 4.5 *Let G be a Lie group and \mathfrak{g} its Lie algebra. Suppose that G has a left invariant connection ∇^G and \mathfrak{g} is endowed with its associated connection $\nabla^{\mathfrak{g}}$. Given M, N semimartingales in \mathfrak{g} we have*

$$\int Ad(e(N))\delta M = \int Ad(e^{G^{\mathfrak{g}}}(N))dM,$$

Proof: First we adopt a basis $\{E_1, \dots, E_n\}$ in \mathfrak{g} and we denote by (x^1, \dots, x^n) the coordinate system associated to this basis. Let M, N be a semimartingales in \mathfrak{g} . We can write $M = M^i E_i$, where $M^i = x^i \circ M$ are real semimartingales, for $i = 1, \dots, n$. By notation,

$$\begin{aligned} \int Ad(e(N))\delta M &= \sum_{i=1}^n \int Ad(e(N))(E_i)\delta M^i \\ &= \sum_{i=1}^n \int Ad(e(N))(E_i)\delta x_i M \\ &= \sum_{i=1}^n \int Ad(e(N))(E_i)dx_i \delta M, \end{aligned}$$

where we use the Itô formula for Stratonovich integral in the last equality. The next step is to apply the relation between the Itô and Stratonovich integrals for functions (see Proposition 2.1). Indeed,

$$\int Ad(e(N))\delta M = \sum_{i=1}^n \int Ad(e(N))(E_i)dx_i d^{\mathfrak{g}}M + \frac{1}{2} \int Ad(e(N))(E_i)\text{Hess}^{\mathfrak{g}}x_i(dM, dM).$$

Using the geometric Itô formula (5) in the second integral of right site we see that

$$\begin{aligned} \int Ad(e(N))\delta M &= \sum_{i=1}^n \int Ad(e(N))(E_i)dM^i - \frac{1}{2} \int Ad(e(N))(E_i)\beta_{x_i}(dM, dM) \\ &\quad + \frac{1}{2} \int Ad(e(N))(E_i)\text{Hess}^{\mathfrak{g}}x_i(dM, dM). \end{aligned}$$

Observing that β_{x_i} and $\text{Hess}^{\mathfrak{g}}x_i$ are equals it follows that

$$\int Ad(e(N))\delta M = \sum_{i=1}^n \int Ad(e(N))(E_i)dM^i = \int Ad(e^{G^{\mathfrak{g}}}(N))dM,$$

and the proof is complete.

Now, we are ready to construct a stochastic Campbell-Hausdorff formulas in the Itô sense.

Theorem 4.6 *Let G be a Lie group and \mathfrak{g} its Lie algebra. Suppose that G has a left invariant connection ∇^G and \mathfrak{g} is endowed with its associated connection $\nabla^{\mathfrak{g}}$. For two semimartingales M, N in \mathfrak{g} holds*

$$e^{G^{\mathfrak{g}}}(M + N) = e^{G^{\mathfrak{g}}} \left(\int Ad(e^{G^{\mathfrak{g}}}(N))dM \right) e^{G^{\mathfrak{g}}}(N),$$

and for two semimartingales X, Y in G we have

$$\mathcal{L}^{G\mathfrak{g}}(X \cdot Y) = \int Ad(Y^{-1})d\mathcal{L}^{G\mathfrak{g}}(X) + \mathcal{L}^{G\mathfrak{g}}(Y).$$

Proof: Let M, N be a semimartingales in \mathfrak{g} . Using the Hausdorff-Campbell formula (9), Lemma 4.5 and Theorem 4.3 we obtain the following sequence of equalities

$$\begin{aligned} e^{G\mathfrak{g}}(M + N) &= e(M + N) = e\left(\int Ad(e(N))\delta M\right) \cdot e(N) \\ &= e^{G\mathfrak{g}}\left(\int Ad(e^{G\mathfrak{g}}(N))dM\right) \cdot e^{G\mathfrak{g}}(N). \end{aligned}$$

The second formula is a direct consequence for the first one.

Corollary 4.7 *For two semimartingales M, N in \mathfrak{g} holds*

1. $e^{G\mathfrak{g}}(M)^{-1} = e\left(-\int Ad(e^{G\mathfrak{g}}(M))dM\right);$
2. $\mathcal{L}^{G\mathfrak{g}}(X^{-1}) = -\int Ad(X)d\mathcal{L}^{G\mathfrak{g}}(X).$

4.1 Skew-symmetric connections on \mathfrak{g}

In this section we adopt the assumption that the connection $\nabla^{\mathfrak{g}}$ on \mathfrak{g} is skew-symmetric. It means that the bilinear form associated to this connection, as in Proposition 4.1, is skew-symmetric. A typical example is the bilinear forms given by multiples of the Lie brackets. One can ask why we do this assumption. The answer is given in the next Lemma.

Lemma 4.8 *Let α a bilinear form from $\mathfrak{g} \times \mathfrak{g}$ into \mathfrak{g} and $\nabla^{\mathfrak{g}}$ the connection on \mathfrak{g} associated to α . Let (x_1, \dots, x_n) a coordinate system on \mathfrak{g} and M a $\nabla^{\mathfrak{g}}$ -martingale on \mathfrak{g} . If α is skew-symmetric, then $M^i = x^i \circ M$ is a real local martingale.*

Proof: It is a direct consequence of Proposition 2.1 and the skew symmetric property of connection $\nabla^{\mathfrak{g}}$.

Here, we call ∇^G of skew-symmetric connection if $\nabla^{\mathfrak{g}}$ is. The assumption of skew-symmetric property is important because with one we show that the product of two martingales on G is a martingale.

Theorem 4.9 *Let G be a Lie group and \mathfrak{g} its Lie algebra. Suppose that G has a left invariant connection ∇^G and \mathfrak{g} is endowed with its associated connection $\nabla^{\mathfrak{g}}$. Suppose that $\nabla^{\mathfrak{g}}$ is skew-symmetric connection. If X, Y are ∇^G -martingales on G , then $X \cdot Y$ is a ∇^G -martingale on G .*

Proof: Let X, Y be ∇^G -martingales in G . By Corollary 3.8, it is sufficient to show that $\mathcal{L}^{G\mathfrak{g}}(X \cdot Y)$ is a $\nabla^{\mathfrak{g}}$ -martingale. From Theorem 4.6 we see that

$$\mathcal{L}^{G\mathfrak{g}}(X \cdot Y) = \int Ad(Y^{-1})d\mathcal{L}^{G\mathfrak{g}}(X) + \mathcal{L}^{G\mathfrak{g}}(Y).$$

Since Y is a ∇^g -martingale, Corollary 3.8 assures that $\mathcal{L}^{G^g}(Y)$ is a ∇^g -martingale. In the other hand, because X is a ∇^g -martingale and ∇^g is skew-symmetric, Lemma 4.8 assures that

$$\int Ad(Y^{-1})d\mathcal{L}^{G^g}(X) = \sum_{i=1}^n \int Ad(Y^{-1})(E_i)d(\mathcal{L}^{G^g}(X))^i$$

is a linear combination, see (11), where each term is a real local martingales. Using Lemma 4.8 again we see that $\int Ad(Y^{-1})d\mathcal{L}^{G^g}(X)$ is a ∇^g -martingale. Consequently, $\mathcal{L}^{G^g}(X \cdot Y)$ is a real local martingale. Thus, by definition, $\mathcal{L}^{G^g}(X \cdot Y)$ is a ∇^g -martingale, and proof is complete.

It is well-known a stochastic characterization of the affine maps (see for instant [8] and [10]). This characterization says that a smooth map $F : (M, \nabla^M) \rightarrow (N, \nabla^N)$ is affine map if and only if F sends ∇^M -martingales to ∇^N -martingales.

Seeing the product on G as application $m : (G \times G, \nabla^G \times \nabla^G) \rightarrow (G, \nabla^G)$ and using Theorem 4.9 we obtain that the product m is affine map.

Corollary 4.10 *Under assumptions of Theorem 4.9, the product $m : G \times G \rightarrow G$ is an affine map.*

In follows we make the following question: given two affine maps with values on a Lie group with a skew-symmetric connection is true that their product is an affine maps? Theorem 4.9 assures that it is true. In fact, for a finite amount of affine maps we have that their product is an affine map.

Theorem 4.11 *Let M_j , $j = 1, \dots, n$, be a differential manifolds with connections ∇^{M_j} and G a Lie group equipped with a left invariant, skew-symmetric connection ∇^G . If $F_j : (M_j, \nabla^{M_j}) \rightarrow (G, \nabla^G)$ are affine maps, then the product map $F_1 \cdot F_2 \cdot \dots \cdot F_n$ from $M_1 \times M_2 \times \dots \times M_n$ into G is an affine map.*

Proof: It is sufficient to prove for $n = 2$. Let $F_1 : (M_1, \nabla^{M_1}) \rightarrow (G, \nabla^G)$ and $F_2 : (M_2, \nabla^{M_2}) \rightarrow (G, \nabla^G)$ be affine maps. Consider the product manifold $M_1 \times M_2$ with connection product $\nabla^{M_1} \times \nabla^{M_2}$. Proposition 3.7 in [10] shows that any $\nabla^{M_1} \times \nabla^{M_2}$ -martingale X can be write as $X = (X_1, X_2)$, where X_i is a ∇^{M_i} -martingale on M_i , $i = 1, 2$. Applying $F_1 \times F_2$ at (X_1, X_2) yields a $\nabla^G \times \nabla^G$ -martingale $(F_1(X_1), F_2(X_2))$ on $G \times G$. According to Corollary 4.10, $F_1(X_1) \cdot F_2(X_2)$ is a ∇^G -martingale on G . It immediately follows that $F_1 \cdot F_2$ is an affine map.

Let $F : M \rightarrow G$ be a smooth map. The map $F^{-1} : M \rightarrow G$ is given by $F^{-1}(x) = (F(x))^{-1}$, $x \in M$. We want to show that F^{-1} is an affine map if F is.

Proposition 4.12 *Let M be a smooth manifold with a connection ∇^M and G a Lie group equipped with a left invariant, skew-symmetric connection ∇^G . If $F : M \rightarrow G$ is an affine map, then the map F^{-1} is an affine map.*

Proof: Let X be a ∇^M -martingale on M . It is sufficient to prove that $\mathcal{L}^{G^g}(F(X)^{-1})$ is a ∇^g -martingale, by Corollary 3.8. From Corollary 4.7 we obtain

$$\mathcal{L}^{G^g}(F(X)^{-1}) = - \int Ad(F(X))d\mathcal{L}^{G^g}(F(X)).$$

F is an affine map, and so $\mathcal{L}^{G\mathfrak{g}}(F(X))$ is a $\nabla^{\mathfrak{g}}$ -martingale. Since $\nabla^{\mathfrak{g}}$ is skew-symmetric connection, $\mathcal{L}^{G\mathfrak{g}}(F(X))$ is a linear combination, see (11), of real local martingales, by Lemma 4.8. Therefore, from Lemma 4.8 we deduce that $\mathcal{L}^{G\mathfrak{g}}(F(X)^{-1})$ is a $\nabla^{\mathfrak{g}}$ -martingale, and this complete the proof.

Example 4.1 *Let M be a smooth manifold with a connection ∇^M and G a Lie group equipped with a left invariant, skew-symmetric connection ∇^G . Let $\mathcal{U}_{aff}(G)$ be the set of the affine maps from M into G . Given two affine maps $f : M \rightarrow G$ and $h : N \rightarrow G$ in $\mathcal{U}_{aff}(G)$ we define the product of them by $(f \cdot h)(x) = f(x) \cdot h(x)$. From Theorem 4.11 we see that $f \cdot h$ is an affine map. Furthermore, $f \cdot h$ is C^∞ because f, h and the product are. In the other side, if f in $\mathcal{U}_{aff}(G)$, then Proposition 4.12 assures that f^{-1} is also an affine map. Therefore $\mathcal{U}_{aff}(G)$ is a subgroup of the Loop group $\mathcal{U}(G) = \{f : M \rightarrow G : f \text{ is a smooth map}\}$. The author do not have a answer when $\mathcal{U}_{aff}G$ is a Lie subgroup of the Banach Lie group $\mathcal{U}(G)$. For a fuller treatment about Loop groups we refer the reader to [16].*

One can ask if the accounts are true for harmonic map instead of affine maps. The answer is true, and this was done by P. Catuogno and P. Ruffino in [6] when G has a bi-invariant metric.

5 Campbell-Hausdorff formulas- second situation

In section 4, we study the case in that $\nabla^{\mathfrak{g}}$ is a skew-symmetric connection to construct the Campbell-Hausdorff formulas. How we showed in Proposition 4.1 ∇^G and $\nabla^{\mathfrak{g}}$ are in association. It is the reason that yields good results. However, in the study of Lie theory we see the Lie algebra \mathfrak{g} as vector space, thus, \mathfrak{g} has a flat connection.

In this section, we assume that $\nabla^{\mathfrak{g}}$ is a flat connection. With this assumption we give up of the good correspondence between ∇^G and $\nabla^{\mathfrak{g}}$ constructed in section 4. To show the Campbell-Hausdorff formulas, the manner that is used to compensate this lack is to introduce a probabilistic condition.

Definition 5.1 *Let M be a smooth manifold and X, Y be two semimartingales on M . We say that X and Y have the null quadratic variation property if for any local coordinates system (x^1, \dots, x^n) on M we have $[X^i, Y^j] = 0$, for $i, j = 1, \dots, n$, where $X^i = x^i \circ X$ and $Y^j = y^j \circ Y$.*

Example 5.1 *Any two independents semimartingales on a smooth manifold M have the null quadratic variation property.*

The null quadratic variation property is a good one because it is preserved by Itô logarithm.

Proposition 5.1 *Let G be a Lie group with a left invariant connection ∇^G and \mathfrak{g} its Lie algebra endowed with flat connection. Given two semimartingales X, Y in G , then X, Y have the null quadratic variation property if, and only if, $\mathcal{L}^{G\mathfrak{g}}(X)$ and $\mathcal{L}^{G\mathfrak{g}}(Y)$ have the null quadratic variation property.*

Proof: Suppose that X, Y are two semimartingales on G such that X, Y have the null quadratic variation property. Thus, for any local coordinates system (U, x^1, \dots, x^n) on G , $[X^i, Y^j] = 0$, where $X^i = x^i \circ X$ and $Y^j = x^j \circ Y$, $i, j = 1, \dots, n$. It is sufficient to prove that $[\mathcal{L}^{G\mathfrak{g}}(X)^\alpha, \mathcal{L}^{G\mathfrak{g}}(Y)^\beta] = 0$ for a global coordinate system (y^1, \dots, y^n) on \mathfrak{g} . By Proposition 2.1,

$$[\mathcal{L}^{G\mathfrak{g}}(X)^\alpha, \mathcal{L}^{G\mathfrak{g}}(Y)^\beta] = \left[\int dy^\alpha d\mathcal{L}^{G\mathfrak{g}}X, \int dy^\beta d\mathcal{L}^{G\mathfrak{g}}Y \right].$$

From definition of the Itô Logarithm we see that

$$[\mathcal{L}^{G\mathfrak{g}}(X)^\alpha, \mathcal{L}^{G\mathfrak{g}}(Y)^\beta] = \left[\int dy^\alpha L_{X*}^{-1}(X) d^G X, \int dy^\beta L_{Y*}^{-1}(Y) d^G Y \right].$$

Applying the definition of the Itô integral (1) yields

$$[\mathcal{L}^{G\mathfrak{g}}(X)^\alpha, \mathcal{L}^{G\mathfrak{g}}(Y)^\beta] = \left[\sum_{l=1}^n \int (dy^\alpha L_{X*}^{-1}(X))^l dX^l, \sum_{k=1}^n \int (dy^\beta L_{Y*}^{-1}(Y))^k dY^k \right].$$

Interchanging the Itô integral with quadratic variation yields

$$[\mathcal{L}^{G\mathfrak{g}}(X)^\alpha, \mathcal{L}^{G\mathfrak{g}}(Y)^\beta] = \sum_{l,k=1}^n \int (dy^\alpha L_{X*}^{-1}(X))^l (dy^\beta L_{Y*}^{-1}(Y))^k d[X^l, Y^k].$$

X, Y have the null quadratic variation property, and so $[\mathcal{L}^{G\mathfrak{g}}(X)^\alpha, \mathcal{L}^{G\mathfrak{g}}(Y)^\beta] = 0$. It gives the null quadratic variation property for $\mathcal{L}^{G\mathfrak{g}}(X)$ and $\mathcal{L}^{G\mathfrak{g}}(Y)$.

Similarly, one can show that if $\mathcal{L}^{G\mathfrak{g}}(X)$ and $\mathcal{L}^{G\mathfrak{g}}(Y)$ have the null quadratic variation property, then X, Y also have the one property.

Further the null quadratic variation property, we must adopt the bi-invariant connections on Lie group to show the Campbell-Hausdorff formula. It get clear in the text of demonstration, when we will need the right invariance property of connection.

Theorem 5.2 *Let G be a Lie group with a bi-invariant connection ∇^G and \mathfrak{g} its Lie algebra endowed with flat connection $\nabla^{\mathfrak{g}}$. Given two semimartingales M, N in \mathfrak{g} , which satisfy the null quadratic variation property, it holds*

$$e^{G\mathfrak{g}}(M + N) = e^{G\mathfrak{g}} \left(\int \text{Ad}(e^{G\mathfrak{g}}(N)) dM \right) e^{G\mathfrak{g}}(N). \quad (12)$$

For two semimartingales X, Y in G , which have the null quadratic variation property, we have

$$\mathcal{L}^{G\mathfrak{g}}(XY) = \int \text{Ad}(Y^{-1}) d\mathcal{L}^{G\mathfrak{g}}(X) + \mathcal{L}^{G\mathfrak{g}}(Y). \quad (13)$$

Proof: We begin introducing the following notation

$$X = e^{G\mathfrak{g}} \left(\int \text{Ad}(e^{G\mathfrak{g}}(N)) dM \right) \quad \text{and} \quad Y = e^{G\mathfrak{g}}(N). \quad (14)$$

The proof of (12) is complete if for each left invariant 1-form θ on G

$$\int \theta d^G(XY) = \int \theta L_{(XY)*}(e) d(M + N).$$

Consider the product on Lie group as the application $m : G \times G \rightarrow G$. Using the geometric Itô formula (5) we get

$$\int \theta d^G(XY) = \int \theta d^G m(X, Y) = \int m^* \theta d^{G \times G}(X, Y) + \frac{1}{2} \int \beta_m^*(d(X, Y), d(X, Y)).$$

We have $\frac{1}{2} \int \beta_m^*(d(X, Y), d(X, Y)) = 0$, because ∇^G is bi-invariant connection and X, Y have the null quadratic variation property. Hence

$$\int \theta d^G(XY) = \int m^* \theta d^{G \times G}(X, Y).$$

From Proposition 3.7 in [10] it may be conclude that

$$\int \theta d^G(XY) = \int R_Y^* \theta d^G(X) + \int L_X^* \theta d^G(Y).$$

Replacing (14) yields

$$\int \theta d^G(XY) = \int R_Y^* \theta L_{X*}(e) Ad(Y) dM + \int L_X^* \theta L_{Y*}(e) dN.$$

A easy computation shows that

$$\int \theta d^G(XY) = \int \theta L_{XY*}(e) dM + \int \theta L_{XY*}(e) dN = \int \theta L_{XY*}(e) d(M + N),$$

and the proof is complete.

The equality (13) is a direct consequence of (12).

Remark 2 A version of Corollary 4.7 is only true for a restrict class of semimartingales. Asking that a semimartingale X has the null quadratic variation yields, by Schwartz principle, a semimartingale that has a second vector field in the first tangent bundle. For example, the Brownian motions do not have the null quadratic variation property (see for instant [8, prop. 18]).

In follows, we generalize the result due to P. Catuogno and P. Ruffino [5] for product of harmonic maps. Before, we introduce a stochastic characterization for harmonic maps. Let (M, g) be a Riemannian manifold and N a manifold with a connection ∇^N . A smooth map $F : M \rightarrow N$ is a harmonic map if and only if it sends g -Brownian motions to ∇^N -martingales.

Theorem 5.3 *Let (M_j, g_j) , $j = 1, \dots, n$, be Riemannian manifolds, G a Lie group with a bi-invariant connection ∇^G and \mathfrak{g} its Lie algebra endowed with a flat connection. If $\phi_j : (M_j, g_j) \rightarrow (G, \nabla^G)$ are harmonic maps, then the product map $\phi_1, \phi_2, \dots, \phi_n$ between $M_1 \times M_2 \times \dots \times M_n$ and G is a harmonic map.*

Proof: It is enough to take $n = 2$. Let $\phi_1 : (M_1, g_1) \rightarrow (G, \nabla^G)$ and $\phi_2 : (M_2, g_2) \rightarrow (G, \nabla^G)$ be harmonic maps. Let B_1 and B_2 two independent Brownian motions in M_1 and M_2 , respectively. Thus (B_1, B_2) is a Brownian motion in the Riemmanian product manifold $M_1 \times M_2$. For Corollary 3.8 is sufficient to show that $\mathcal{L}^{G\mathfrak{g}}(\phi_1(B_1) \cdot \phi_2(B_2))$ is a local martingale. B_1 and B_2 are

independent, and consequently they have the null quadratic variation. Theorem 5.2 now assures that

$$\mathcal{L}^{G\mathfrak{g}}(\phi_1(B_1) \cdot \phi_2(B_2)) = \int Ad(\phi_2(B_2)^{-1})d\mathcal{L}^{G\mathfrak{g}}(\phi_1(B_1)) + \mathcal{L}^{G\mathfrak{g}}(\phi_2(B_2)). \quad (15)$$

Since ϕ_i , $i = 1, 2$, are harmonic maps, $i = 1, 2$, $\phi_1(B_1)$ and $\phi_2(B_2)$ are ∇^G -martingales on G . Therefore $\mathcal{L}^{G\mathfrak{g}}(\phi_1(B_1))$ and $\mathcal{L}^{G\mathfrak{g}}(\phi_2(B_2))$ are local martingales in \mathfrak{g} by Corollary 3.8. Being $\nabla^{\mathfrak{g}}$ a flat connection on G , the right side of (15) is a sum of local martingales. Consequently, $\mathcal{L}^{G\mathfrak{g}}(\phi_1(B_1) \cdot \phi_2(B_2))$ is a local martingale, and the proof is complete.

Example 5.2 Let G be a Lie group equipped with a bi-invariant connection ∇^G and \mathfrak{g} its Lie algebra endowed with flat connection. Let γ_i be ∇^G -geodesics on G , $i = 1, \dots, n$. A map $f : (\mathbb{R}^n, <, >) \rightarrow G$ defined by

$$f(t_1, t_2, \dots, t_n) = \gamma_1(t_1) \cdot \gamma_1(t_2) \cdot \dots \cdot \gamma_n(t_n).$$

is harmonic. It is a direct consequence of Theorem 5.3. Indeed, it is sufficient to see any geodesic γ_i , $i = 1, \dots, n$ as a harmonic map.

In particular, assume that G has a bi-invariant metric. Choose n vectors $X_1, X_2, \dots, X_n \in \mathfrak{g}$ such that $\exp(t_1 X_1), \exp(t_2 X_2), \dots, \exp(t_n X_n)$ are geodesics (see for instance [7]). According to the facts above, $\exp(t_1 X_1) \cdot \exp(t_2 X_2) \cdot \dots \cdot \exp(t_n X_n)$ is a harmonic map. This example is also founded in [7] and [5].

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